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# The Constituent Quark as a Topological Soliton \*

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## Abstract

Recently it was proposed that the constituent quark is a topological soliton. I investigate this soliton, calculating its mass, radius, magnetic moment, color magnetic moment, and spin structure function. Within the approximations used, the magnetic moments and spin structure function cannot simultaneously be made to agree with the constituent quark model. Some discussion of what to expect from better approximations is included.

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# 1 Introduction

Why does the constituent quark model work? This paper examines one possible answer to that question.

The constituent quark model is puzzling because all the clues from QCD and current algebra [1] indicate that the nucleon is made of a swarm of gluons and nearly massless quarks and anti-quarks, all interacting strongly. According to the constituent quark model, however, the nucleon is composed of nothing but three massive quarks, interacting weakly. The model works well: baryon [2, 3] and heavy meson [4] masses can be fitted to within a few percent, and most baryon magnetic moments [1, 3] to within 30%.

One explanation for the success of this model is that the nucleon really does contain three weakly interacting components. In this picture, the nearly massless “current quarks” are fundamental particles, and through their strong interactions each is able to draw around itself a cloak of virtual gluons and quark–anti-quark pairs, resulting in the collective excitation called a “constituent quark”. The constituent quark has the same spin and flavor as the original current quark, but is heavier and less strongly interacting.

To explain how the constituent quark might arise as a collective excitation, two models have been proposed. The first is Manohar and Georgi’s chiral quark model [5] (which is closely related to the Nambu–Jona-Lasinio model [6]). In this model the current quark increases its mass by coupling to the quark condensate that forms when chiral symmetry is broken. The resulting constituent quark then automatically has the same spin and flavor as the original current quark.

The second model is the quark soliton model proposed by Kaplan [7]. It is based on a very simple and appealing idea: the quark condensate may undergo rotations in color space as well as flavor space. The flavor rotations give rise to the usual non-linear sigma model Lagrangian used in pion physics. One form of this Lagrangian, the Skyrme Lagrangian [8, 9, 10], will support a topological soliton which is a model of the nucleon. Similarly, the color rotations of the condensate can be described by a Lagrangian which also supports a soliton. This soliton is a candidate for the constituent quark. The topological properties of this winding number 1 soliton ensure that it has spin 1/2 and baryon number 1/3, just as the original current quark.

This soliton has been analyzed in two dimensions [11]. In four dimensions, its mass and radius have been computed [12], and it was found that either the soliton's mass or its radius (or both) must be larger than expected for the constituent quark. However, Ref. [12] argues that this is not a fatal flaw in the model.

In this paper I have used a different technique to study the mass and radius, and have reached similar conclusions. I have gone on to evaluate the soliton's magnetic moment, color magnetic moment, and spin structure function. Within the approximations used, the spin structure function and the magnetic moments cannot both be fitted simultaneously in this model. Some speculation is offered on what to expect from better approximations.

The main purpose of this paper is to show how the static properties of the soliton may be evaluated, and to test whether the soliton's properties are compatible with the constituent quark's. Section 2 introduces the Lagrangian and the soliton. Section 3 shows how I extract the static properties of the soliton. These properties are then computed and the results given in Section 4. The paper closes with a short discussion in Section 5.

## 2 The Soliton

In the quark soliton model, the quark condensate can undergo rotations in color space as well as in flavor space. The color degrees of freedom are parametrized by  $U = e^{2i\Pi^a T^a/f}$ . The capital  $\Pi^a$  is used to distinguish this field from the ordinary pion field  $\pi^a$ ; the  $T^a$  are the generators of color SU(3); the constant  $f$  is analogous to the pion decay constant  $f_\pi$ . With  $\tilde{R}_\mu \equiv U^\dagger D_\mu U$ , Kaplan [7] proposed the following one-flavor Lagrangian (which could be extended to more flavors later) :

$$\begin{aligned} \mathcal{L}[U, A_\mu] = & -\frac{1}{4g^2} G_a^{\mu\nu} G_{\mu\nu}^a - \frac{f^2}{4} \text{tr}(\tilde{R}_\mu \tilde{R}^\mu) + \frac{1}{32e^2} \text{tr}([\tilde{R}_\mu, \tilde{R}_\nu]^2) \\ & + \frac{2}{3} f^4 \nu^2 \text{tr}(T_a U T_a U^\dagger) + n \mathcal{L}_{WZ} \end{aligned} \quad (1)$$

This Lagrangian is patterned after the Skyrme Lagrangian [8, 9, 10]. The first term gives kinetic energy to the gluons; the second gives kinetic energy to the

chiral field  $U$ . The third term, called the “Skyrme term”, is introduced to stabilize the soliton solution. If this term is absent from the ordinary (ungauged) Skyrme Lagrangian, then the soliton shrinks to zero size. The fourth term breaks color  $SU(3)_L \times SU(3)_R$  symmetry, and consequently gives mass to the (as yet undiscovered)  $\Pi$  particles. This term is necessary because QCD interactions, whose low energy behaviour this Lagrangian is intended to model, explicitly break *color* (not flavor!)  $SU(3)_L \times SU(3)_R$  symmetry. The last term of Eq.(1) is the Wess–Zumino term [13, 14], which takes anomalies into account.

Each of the first four terms is multiplied by an arbitrary constant, to be fitted phenomenologically.  $g$ ,  $e$ , and  $\nu$  are dimensionless;  $f$  has the dimensions of mass. The mass of the  $\Pi$  particle is equal to  $2\nu f$ . The integer coefficient of the Wess–Zumino–Witten anomaly term is  $n = 1$ , as opposed to  $n = N_c = 3$  in the Skyrme Lagrangian. The soliton solutions of this Lagrangian have baryon number  $n/N_c$  [7], and upon rotation by  $2\pi$ , the soliton will acquire a phase  $(-1)^n$  [13]. Therefore the soliton is a fermion with baryon number  $1/3$ . This soliton, which Kaplan calls a “qualiton”, is a candidate for the constituent quark.

The construction of the soliton from the Lagrangian (1) proceeds in three steps which will be sketched below (see [7] for more details). Step one: construct the classical solution. The field  $U$  takes on the “hedgehog” ansatz form:

$$U_{cl} = e^{iF(r)\hat{\mathbf{r}}\cdot\boldsymbol{\tau}} \quad (2)$$

where the  $\tau^i$  are the Pauli matrices embedded in color  $SU(3)$ . In this ansatz  $F(0) = \pi$  and  $F(\infty) = 0$ . The gauge field is given by

$$A_{i\,cl} = -A_{cl}^i = i\frac{\gamma(r)}{2r}\epsilon_{ijk}\hat{r}^j\tau^k, \quad i = 1, 2, 3 \quad (3)$$

Throughout this paper an anti-hermitian gauge field is used:  $D_\mu = \partial_\mu + A_\mu$ . The profile functions  $F$  and  $\gamma$  can be determined by minimizing the soliton’s classical mass,  $m_{cl} = -\mathcal{L}[U_{cl}, A_{i\,cl}]$ . The resulting Euler–Lagrange equations can be solved numerically [15]. It is convenient to use the dimensionless variable  $\tilde{r} = 2fr$ , since the Euler-Lagrange equations that determine  $F(\tilde{r})$  and  $\gamma(\tilde{r})$  do not depend on the parameter  $f$ . Fig. 1 shows the profiles  $F(\tilde{r})$  and  $\gamma(\tilde{r})$  for  $1/e = 0$ ,  $g = 12.4$ , and  $\nu = 237$ . These values were chosen for two reasons: first, the resulting soliton has the same spin structure function as expected for the

constituent quark (see below). Second, this demonstrates that the soliton can be stable even when the Skyrme term is absent ( $1/e = 0$ ). Evidently, the gauge field is sufficient to stabilize the soliton. (A similar feature has been seen in the Skyrme Lagrangian, where the Skyrme solution itself is stable in the absence of the Skyrme term as long as the  $\rho$ -meson gauge field is present[16].)

Step two of constructing the soliton: make it rotate. This is done by conjugating the field  $U_{cl}$  by a matrix  $\Omega$ :

$$U = \Omega U_{cl} \Omega^\dagger \quad (4)$$

If the gauge field were absent,  $\Omega$  would be a function of time but not of space. Since the gauge field is present, however,  $\Omega$  must depend on  $r$  as well as  $t$ . This can be understood as follows: the gauge field rotates also:

$$\mathbf{A} = \Omega \mathbf{A}_{cl} \Omega^\dagger - \Omega \nabla \Omega^\dagger \quad (5)$$

The rotation of  $U$  and  $\mathbf{A}$  generates a charge density

$$j_0^a = i \frac{f^2}{2} \text{tr}[(U^\dagger T^a U - T^a) \tilde{R}_0] - \frac{i}{8e^2} \text{tr}\{[(U^\dagger T^a U - T^a), \tilde{R}^\nu][\tilde{R}_0, \tilde{R}_\nu]\} \quad (6)$$

The color electric fields must be given by the rotating gauge fields of Eq. (5), and must also satisfy Gauss's Law with the charge of Eq. (6). This constrains  $\Omega$  to satisfy certain differential equations given in Ref. [7]. For now it is enough to know that  $\Omega$  can be parametrized by three functions  $\omega_1(r)$ ,  $\omega_2(r)$ , and  $\omega_3(r)$ . These functions are shown in Fig. 2 for the same set of input parameters as in Fig. 1.  $\omega_1$  and  $\omega_3$  are closely analogous to ordinary angular velocity, and they are smaller for smaller  $r$ . That is, the soliton must rotate more slowly in the middle than on the outside because otherwise it generates too much charge to be consistent with Gauss's Law.

As  $r \rightarrow \infty$ , the matrix  $\Omega(r, t)$  becomes equal to a matrix  $W(t)$ . The Lagrangian (1) can be rewritten in terms of this matrix as [7]:

$$L = -m_{cl} + \frac{I_1}{2} \sum_{m=1,2,3} (iW^\dagger \dot{W})_m^2 + \frac{I_2}{2} \sum_{\alpha=4\dots 7} (iW^\dagger \dot{W})_\alpha^2 + \frac{1}{\sqrt{12}} (iW^\dagger \dot{W})_8 \quad (7)$$

Here  $(W^\dagger \dot{W})_a \equiv 2 \text{tr}(T_a W^\dagger \dot{W})$ . The last term in this Lagrangian comes from the Wess-Zumino term. The moments of inertia  $I_1$  and  $I_2$  are coefficients that can be computed once  $F$ ,  $\gamma$ ,  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  are known.

The Lagrangian (7) describes a top spinning in SU(3) space. When the Hamiltonian is constructed, the canonical momenta  $P_a$  are used. Explicitly [10],  $P_a = I_1(iW^\dagger \dot{W})_a$  for  $a = 1, 2, 3$  and  $P_a = I_2(iW^\dagger \dot{W})_a$  for  $a = 4, \dots, 7$ .  $P_a$  is equal to the ordinary angular momentum  $\Lambda_a$  for  $a = 1, 2, 3$ . Since the Wess–Zumino term only contributes one power of  $(W^\dagger \dot{W})_8$  to the Lagrangian (7),  $P_8$  will not have an equation of motion but rather an equation of constraint:  $P_8 = 1/\sqrt{12}$ .

Step three in building the soliton: quantize it. Here  $W$  and  $P_a$  are no longer treated as ordinary matrices, but as operators on a Hilbert space. The total mass of the soliton is given by the resulting energy eigenvalues,

$$E = M_{TOT} = m_{cl} + j(j+1)\left(\frac{1}{2I_1} - \frac{1}{2I_2}\right) + \frac{1}{2I_2}\left(C_2 - \frac{1}{12}\right) \quad (8)$$

where  $j$  is the spin quantum number and  $C_2$  is the color SU(3) casimir. Since we are interested in a spin-1/2, color triplet particle,  $j = 1/2$  and  $C_2 = 4/3$ . Then

$$M_{TOT} = m_{cl} + \frac{3}{8I_1} + \frac{1}{4I_2} \quad (9)$$

The soliton is described by a state  $|q, \sigma\rangle$  whose wave functions are given by the SU(3) Wigner D-functions in the triplet representation [10, 17, 18]:

$$\psi_{q,\sigma}(W) = \langle W | q, \sigma \rangle = \sqrt{3} D_{q,\sigma}^{(3)}(W) \quad (10)$$

Here  $q$  and  $\sigma$  are SU(3) indices:  $q = (I, I_3, Y)$  gives the color isospin and hypercharge of the particle, and  $\sigma = (s, -m_s, 1/3)$  gives the spin. The last entry is constrained to be 1/3 by the Wess–Zumino term.

There is one point worth mentioning now. In the above procedure, the functions  $F$  and  $\gamma$  are determined by minimizing the classical mass; they are then used to calculate the moments of inertia  $I_1$  and  $I_2$ . This is called the “semi-classical” approach. A more exact procedure [19] is to view the total mass as a functional of  $F$  and  $\gamma$ ,

$$M_{TOT}[F, \gamma] = m_{cl}[F, \gamma] + \frac{3}{8I_1[F, \gamma]} + \frac{1}{4I_2[F, \gamma]} \quad (11)$$

and then find those functions  $F$  and  $\gamma$  which minimize the total mass, not just the classical mass. The resulting integro-differential equations have never been worked out.

In the Skyrme model the semi-classical approach is sufficient because  $m_{cl}$  is of order  $N_c$  and the moments of inertia are also of order  $N_c$ . Therefore the rotational energy does not contribute much to the total energy, and the error made in the semi-classical approximation is small. However, no such  $N_c$ -counting argument exists for the qualiton, and there is no guarantee that the semi-classical approach is enough. Still, it is worth trying.

Having constructed the soliton, then, the question is whether the four parameters  $f$ ,  $e$ ,  $g$ , and  $\nu$  can be adjusted to give realistic values for the static properties of the constituent up and down quarks.

### 3 Properties of the Soliton

The static properties of the soliton are discussed in this section. For each observable, I first state what is expected from the static quark model, and then describe how to extract this quantity from the soliton model. The results are then used in Section 4 for numerical computations.

#### Mass and Radius

The constituent quark mass is typically taken to be  $\approx 350$  MeV. For definiteness, I will take Quigg's value,  $m = 362$  MeV [3]. The radius of the constituent quark should be less than the radius of the nucleon. The isoscalar rms radius of the nucleon is  $r_{rms} = 0.72$  fm. Therefore, in units where  $\hbar = c = 1$ , there is an upper bound on the dimensionless quantity  $m \cdot r_{rms}$  for the constituent quark:

$$mr_{rms} \leq 1.3 \tag{12}$$

In the soliton model,  $r_{rms}$  can be computed using the singlet part of the anomalous Wess–Zumino current:

$$r_{rms}^2 = \int d^3r \, r^2 \, J_{WZ}^0$$

where [14]

$$J_{WZ}^\mu = \frac{1}{48\pi^2} \epsilon^{\mu\alpha\beta\gamma} \text{tr} [2\tilde{R}_\alpha \tilde{R}_\beta \tilde{R}_\gamma - 3F_{\alpha\beta}(\tilde{R}_\gamma + \tilde{L}_\gamma)]$$

Here  $\tilde{R}_\mu = U^\dagger D_\mu U$  and  $\tilde{L}_\mu = (D_\mu U)U^\dagger$ . The convention  $\epsilon_{0123} = -\epsilon^{0123} = 1$  is used. Using Eqns. (2) - (5) (cf. [7]),

$$r_{rms}^2 = \frac{1}{\pi} \frac{1}{(2ef)^2} \int \tilde{r} d\tilde{r} [2F - (1 + \gamma)^2 \sin 2F] \quad (13)$$

Given the radius from Eq.(13) and the mass as calculated from Eq. (9), can the qualiton satisfy the inequality (12)?

To answer this question, it is easiest to look first in the limit where  $g \rightarrow 0$  and  $\nu \rightarrow 0$ , because the result is identical to the ordinary SU(3) Skyrmeion with  $N_c = 1$ . In this model the total mass is

$$\begin{aligned} M &= m_{cl} + \frac{3}{8I_1} + \frac{1}{4I_2} \\ &= \hat{m} \frac{2f}{e} + \frac{3}{8} \left( \frac{2fe^3}{\hat{I}_1} \right) + \frac{1}{4} \left( \frac{2fe^3}{\hat{I}_2} \right) \end{aligned}$$

where  $\hat{m}$ ,  $\hat{I}_1$ , and  $\hat{I}_2$  can be calculated numerically:  $\hat{m} = 36.5$ ,  $\hat{I}_1 = 106.6$ ,  $\hat{I}_2 = 40.6$ . (Ref. [10] gets similar values.) The isoscalar rms radius is

$$r_{rms} = \frac{\hat{r}_{rms}}{2fe}$$

with  $\hat{r}_{rms} = 2.12$  (cf. [9]). Therefore,

$$Mr_{rms} = \hat{r}_{rms} \left[ \frac{\hat{m}}{e^2} + \left( \frac{3}{8\hat{I}_1} + \frac{1}{4\hat{I}_2} \right) e^2 \right]$$

Differentiating the above equation with respect to  $e$  and setting it equal to zero reveals that, when  $g = \nu = 0$ ,

$$Mr_{rms} \geq 2.52 \quad (14)$$

with the minimum occurring at  $e=7.84$ . Numerically, it is found that the full soliton also has a minimum  $(Mr_{rms}) = 2.52$  at this ‘‘Skyrme’’ configuration ( $e = 7.84$ ,  $\nu = 0$ , and  $g = 0$ );  $(Mr_{rms})$  only increases when  $g$  and  $\nu$  move away from 0.

Therefore it is not possible for the soliton to satisfy the inequality (12). Ref. [12] uses a different technique but comes to the same conclusion: the radius or the mass of the soliton is larger than expected for a constituent quark. Ref.



[12] explores whether a large radius is a serious flaw. The problem with this approach is that it requires the confining force to be so strong that it contracts the quark to roughly half its original size. This goes against the spirit of the constituent quark model, in which the quarks are weakly interacting inside the hadron.

The alternate possibility is that the mass is large. At first it might seem that this, too, would violate the principles of the constituent quark model, since the binding energy per quark would be at least half the constituent mass. This would require the quarks to be very strongly interacting as well. However, the concept of binding energy does not exist in confining theories. In fact, until details of the inter-quark forces are included, the relationship between the mass of the constituents and the mass of the nucleon cannot be determined. Since these details have not yet been worked out, the question of the excessive mass cannot be addressed in this paper.

## Magnetic Moment

The magnetic moment of a particle with charge  $q$  and spin  $\mathbf{S}$  can be written

$$\boldsymbol{\mu} = q\beta\mathbf{S} \quad (15)$$

where  $\beta$  is a parameter with the dimensions of length. In the constituent quark model (where the quarks have a Dirac  $g$ -factor of two),  $\beta = 1/m$ . Using  $m = 362$  MeV, the value of  $\beta$  is 0.544 fm.

In the soliton model, the magnetic moment is given by

$$\boldsymbol{\mu} = q\frac{1}{2}\int d^3x \mathbf{r} \times \mathbf{J}_{WZ} \quad (16)$$

It is easiest to work in the gauge where  $\mathbf{A}_i = \mathbf{A}_{icl}$ ,  $A_0 = \Omega^\dagger \dot{\Omega}$ , and  $U = U_{cl}$ . Then,

$$J_{WZ}^i = -\frac{1}{8\pi^2}\epsilon^{ijk}\partial_k tr[A_0 U^\dagger \partial_j U + A_0 \partial_j U U^\dagger + A_0 U^\dagger A_j U - A_0 U A_j U^\dagger] \quad (17)$$

$A_0$  can be written in terms of the angular momentum  $\mathbf{\Lambda}$ :

$$\begin{aligned} A_0 = \Omega^\dagger \dot{\Omega} &= \frac{-i}{2I_1}[(\omega_1 + \omega_2)\mathbf{\Lambda} \cdot \boldsymbol{\tau} - \omega_2(\hat{\mathbf{r}} \cdot \mathbf{\Lambda})\hat{\mathbf{r}} \cdot \boldsymbol{\tau}] \\ &+ \text{terms} \propto \lambda_4 \dots \lambda_8 \end{aligned}$$

The extra terms proportional to  $\lambda_4 \dots \lambda_8$  do not contribute to the trace in Eq. (17). The resulting magnetic moment is

$$\begin{aligned} \boldsymbol{\mu} = & \lim_{R \rightarrow \infty} -\frac{q}{3\pi} \frac{\boldsymbol{\Lambda}}{I_1} \left\{ \int_0^R r^2 dr [\omega_1 F' + \frac{1}{r}(\omega_1 + \omega_2)(1 + \gamma) \sin 2F] \right. \\ & \left. + [r^3 \omega_1 F' + \frac{1}{2} r^2 (\omega_1 + \omega_2)(1 + \gamma) \sin 2F]_{r=R} \right\} \end{aligned} \quad (18)$$

In Section 4 this formula is used for the numerical computation of  $\beta$  for the soliton.

### Color Magnetic Moment

In the constituent quark model, the hyperfine mass splitting of the hadrons is given by the interaction of the color magnetic moments of the constituents:

$$\Delta E_{hfs} = -\frac{2}{3} |\psi(0)|^2 \sum_{i < j} \langle n | \boldsymbol{\mu}_{(i)}^a \cdot \boldsymbol{\mu}_{(j)}^a | n \rangle$$

where the sum is over the quarks  $i$  and  $j$  in the nucleon  $|n\rangle$ . The color magnetic moment can be defined by

$$\boldsymbol{\mu}^a = \mu_c \mathbf{S} \lambda^a \quad (19)$$

where the  $\lambda^a$  are the Gell-Mann matrices and  $\mu_c$  is a parameter with the dimensions of length. In the constituent quark model,  $\mu_c = g/2m$ . Using  $\alpha_s = g^2/4\pi = 0.4$  and  $m = 362$  MeV [3], the value of  $\mu_c$  is 0.610 fm. The ratio of the constituent quark's color magnetic moment to its magnetic moment is proportional to  $\mu_c/\beta = 1.12$ .

The qualiton also has a color magnetic moment, which can be extracted from the asymptotic behaviour of its B-field. The standard dipole form for  $\mathbf{B}$ , using the normalization of Eq. (1), is

$$B_a^i = \frac{g\mu_a^j}{4\pi} (3\hat{r}_i \hat{r}_j - \delta_{ij}) \frac{1}{r^3} \quad (20)$$

The B-field of the qualiton will turn out to have a similar form at large  $r$ , so the coefficient  $\mu_a^j$  can easily be determined.

At large radius, the B-field of the classical soliton is

$$B_{a\,cl}^i = -r_B(3\hat{r}_i\hat{r}_a - \delta_{ia})\frac{1}{r^3} \quad (21)$$

The constant  $r_B$  is determined by the numerical solution for  $\gamma : \lim_{r \rightarrow \infty} \gamma(r) = -r_B/r$ . The B-field of the quantized soliton can now be calculated.

Under the quantization procedure,  $\mathbf{B}$  becomes an operator  $\hat{\mathbf{B}}$  rather than just a matrix. At large radius,  $\hat{\mathbf{B}} = W\mathbf{B}_{cl}W^\dagger$ . The expectation value of  $\hat{\mathbf{B}}$  with respect to the quark soliton state  $|q, \sigma\rangle$  is

$$\mathbf{B} = \langle q, \sigma | \hat{\mathbf{B}} | q, \sigma \rangle = \langle q, \sigma | W\mathbf{B}_{cl}W^\dagger | q, \sigma \rangle$$

Using  $B^i = B_m^i T^m$ ,

$$B_n^i = 2 \langle q, \sigma | \text{tr}[T^n W T^m W^\dagger] | q, \sigma \rangle B_{m\,cl}^i \quad (22)$$

In order to compute the above matrix element, we can use the wavefunctions given in Section 2, Clebsch–Gordan coefficients [20], and the identity [18]

$$\langle W | \text{tr}[T_n W T_m W^\dagger] | W \rangle = \frac{1}{2} D_{nm}^{(8)}(W)$$

The result is

$$\langle q, \sigma | \text{Tr}[T^n W T^m W^\dagger] | q, \sigma \rangle = -\frac{3}{32} \langle q, \sigma | \sigma^m \lambda_n | q, \sigma \rangle \quad (23)$$

Combining this equation with (22) and (21) gives

$$B_n^i = \frac{3r_B}{16} \langle q, \sigma | \sigma^m \lambda_n | q, \sigma \rangle (3\hat{r}_i\hat{r}_m - \delta_{im})\frac{1}{r^3} \quad (24)$$

Therefore, using Eq. (20),

$$\boldsymbol{\mu}_a = \frac{4\pi}{g} \frac{3r_B}{16} \boldsymbol{\sigma} \lambda_a \quad (25)$$

Using  $\mathbf{S} = \boldsymbol{\sigma}/2$  and Eq. (19), we find that  $\mu_c = (4\pi/g)(3r_B/8)$ . This information is used in Section 4.

The above procedure is sufficient to determine the color magnetic moment, but there is another way to evaluate it which parallels the calculation of the ordinary magnetic moment. Testing whether these two methods agree serves as a useful check on the numerical computations.

The color magnetic moment should be given by

$$\boldsymbol{\mu}_a = \frac{g}{2} \int d^3r \mathbf{r} \times \mathbf{J}_a$$

where  $J_a^\mu$  is the current which couples to the gauge field  $A_a^\mu$ . This integral has already been worked out in the SU(2) case [9] for  $\nu, \alpha_s \rightarrow 0$ . It is unchanged for SU(3), and the result is

$$\mu_a^i = -g I_1 \text{tr}[T^a W T^i W^\dagger] \quad (26)$$

where the trace is to be taken as a matrix element between quark states, as above. Combining this with Eq. (25) gives

$$r_B = \frac{\alpha_s I_1}{2} \quad (\alpha_s, \nu \ll 1)$$

This expression is indeed satisfied by the numerical computations of  $r_B$  and  $I_1$  when  $\alpha_s$  and  $\nu$  are small.

## Spin Structure

The spin structure function of the constituent quark is not as well established as the previous properties. In fact, it is not obvious from the recent spin structure experiments whether the data are even consistent with the constituent quark model at all. They turn out to be consistent, but some explanation is required.

The Fourier transform of the constituent quark spin structure function is defined as follows: for a single spin-up quark  $|q \uparrow\rangle$  and the field  $\psi$  which annihilates it,

$$g_q(\mathbf{r}) = \langle q \uparrow | \bar{\psi}(\mathbf{r}) \gamma_z \gamma_5 \psi(\mathbf{r}) | q \uparrow \rangle$$

I will call the integral over all space of  $g_q$  the “spin content”  $s_q$ :

$$s_q = \int d^3r g_q(\mathbf{r})$$

In the non-relativistic limit one would expect  $s_q = \sigma_z = 1$ . As we will see below, the recently measured spin structure functions of the neutron and proton will force us to change these expectations.

To begin, look at the nucleon. The contribution of the up quark spin to a spin-up proton is:

$$\Delta u = \int d^3r \langle p \uparrow | \bar{u} \gamma_z \gamma_5 u | p \uparrow \rangle \quad (27)$$

If the up quark is non-relativistic,  $\Delta u$  is just equal to the number of up quarks with spin parallel to the proton's spin, minus the number of up quarks with spin anti-parallel.  $\Delta d$  and  $\Delta s$  are similarly defined. Several relations exist between these quantities:

$$\Delta u - \Delta d = g_A = 1.2573 \pm .0028 \quad (28)$$

is used in the Bjorken sum rule [21], and

$$\Delta s - \Delta d = D - F = .328 \pm .019 \quad (29)$$

results from an analysis of semileptonic hyperon decay [22]. Combining both of the above two equations with a third equation would give three equations and three unknowns, and so  $\Delta u$ ,  $\Delta d$ , and  $\Delta s$  could all be determined. Actually, any one of several equations could be chosen as the third equation in this procedure. The EMC experiment [23] gives

$$\frac{1}{2} \left( \frac{4}{9} \Delta u + \frac{1}{9} \Delta d + \frac{1}{9} \Delta s \right) = 0.126 \pm 0.02 \quad (30)$$

The result of the E142 experiment [24] is

$$\frac{1}{2} \left( \frac{1}{9} \Delta u + \frac{4}{9} \Delta d + \frac{1}{9} \Delta s \right) = -0.022 \pm 0.011 \quad (31)$$

and the SMC experiment [25] gives

$$\frac{1}{4} \left( \frac{5}{9} \Delta u + \frac{5}{9} \Delta d + \frac{2}{9} \Delta s \right) = 0.023 \pm 0.025 \quad (32)$$

Equations (28) and (29) can be combined with either Eq. (30), (31), or (32). The results of all three possibilities are given in the first three lines of Table 1.

The various results do not agree. This discrepancy has inspired some discussion [26, 27]. Ellis and Karliner [26] show, for example, that the three experiments do agree as long as perturbative QCD corrections are taken into account. The results of their analysis, including these corrections, is shown in the fourth line of Table 1. The QCD corrections will not exactly apply to the soliton

model, but they are listed to give an idea of the range of values currently under discussion.

How does all of this relate to the constituent quark? In the constituent quark model, the matrix element in Eq.(27) can be related to the helicity of the individual quarks and to the polarization of the gluons [28] present in the proton:

$$\begin{aligned}\Delta u &= \frac{4}{3} \int d^3r \langle q \uparrow | \bar{\psi} \gamma_z \gamma_5 \psi | q \uparrow \rangle - \frac{\alpha_s}{2\pi} \Delta g \\ \Delta d &= -\frac{1}{3} \int d^3r \langle q \uparrow | \bar{\psi} \gamma_z \gamma_5 \psi | q \uparrow \rangle - \frac{\alpha_s}{2\pi} \Delta g \\ \Delta s &= 0 - \frac{\alpha_s}{2\pi} \Delta g\end{aligned}$$

As before,  $|q \uparrow\rangle$  is a single quark state (of either up or down flavor), and  $\psi$  annihilates that quark. The gluon contribution  $\Delta g$  must be included, because the current appearing in Eq. (27) can interact via a quark loop with the gluon sea, which may be polarized. The prefactors  $4/3$  and  $-1/3$  come from the constituent quark model wavefunctions [3].

In the naive constituent quark model  $\int d^3r \langle q \uparrow | \bar{\psi} \gamma_z \gamma_5 \psi | q \uparrow \rangle \equiv s_q = 1$  and  $\Delta g = 0$ . This results in line 5 of Table 1, which does not agree well with experiment. However, if for some reason  $s_q$  turns out to equal  $3/4$ , then  $g_A = \Delta u - \Delta d = 5/4$ , which is very close to the experimental value. If in addition  $(\alpha_s/2\pi)\Delta g = 0.2$ , then the values of  $\Delta u$ ,  $\Delta d$ , and  $\Delta s$  more or less agree with experiment. These values are shown in the last line of Table 1. (The same values have been used in the context of a relativistic quark model in Ref. [29].) Therefore the spin content of the quark soliton will be assumed to be

$$s_q \equiv \int d^3r \langle q \uparrow | j_{(5)}^3(r) | q \uparrow \rangle = \frac{3}{4} \quad (33)$$

In the soliton model, the color singlet axial-vector current  $j_{(5)}^\mu$  arises only from the Wess–Zumino anomaly term. In order to compute this current, it is necessary to start from the general Wess–Zumino Lagrangian which includes both left- and right-handed fields [14]. Then

$$\begin{aligned}j_{(5)}^\mu &= \frac{1}{i} \left( \frac{\partial \mathcal{L}_{WZ}}{\partial A_\mu^R} - \frac{\partial \mathcal{L}_{WZ}}{\partial A_\mu^L} \right)_{A^L=A^R=A} \\ &= -\frac{1}{48\pi^2} \epsilon^{\mu\alpha\beta\gamma} \text{tr}[F_{\alpha\beta}(\tilde{R}_\gamma - \tilde{L}_\gamma)]\end{aligned} \quad (34)$$

and

$$\begin{aligned} \int d^3r \mathbf{j}_{(5)} = & \frac{\mathbf{\Lambda}}{18\pi^2 I_1} \int d^3r \frac{\sin^2 F}{r} \left[ (1 + \gamma)(\omega'_1 + \omega'_2 + \frac{\omega_2}{r}) \right. \\ & \left. - \omega_1 \gamma (1 + \gamma) \frac{1}{r} - \gamma'(\omega_1 + \omega_2) \right] \end{aligned} \quad (35)$$

where again  $\mathbf{\Lambda}$  is the angular momentum of the soliton. From here, the matrix element required in Eq. (33) can be obtained easily.

## 4 Numerical Results

The static properties of the quark soliton can now be computed. It is easiest to look at dimensionless quantities, since these quantities are determined only by the three dimensionless parameters  $e$ ,  $g$ , and  $\nu$ . The fourth parameter  $f$  determines the overall scale, and can be fixed later.

The constituent quark can be described by the following two dimensionless quantities: the ratio of its color magnetic moment to its magnetic moment,  $\mu_c/\beta = 1.12$ , and its spin content,  $s_q = 0.75$ . Requiring that the soliton's ratio  $\mu_c/\beta$  take on the physical value of 1.12 will constrain the permissible values of  $e$ ,  $g$ , and  $\nu$  to lie on a two-dimensional surface within the three-dimensional parameter space. Alternatively, requiring that the spin content  $s_q$  achieve its physical value of 0.75 will define a different surface within the parameter space. In general, these two surfaces will intersect to form a (one-dimensional) curve. This curve is the family of points where the soliton is a good model of the constituent quark. Unfortunately, I have found that these two surfaces do not intersect.

To begin searching the parameter space for points appropriate to a constituent quark, one might first try the point suggested by Ref. [12] ( $e = 5.7$ ,  $\alpha_s = 0.28$ ,  $\nu = 0.36$ ). However, this gives  $\mu_c/\beta = 1.94$  and  $s_q = 0.0041$  (compared to the experimental values of 1.12 and 0.75).

The parameter space can be searched using the methods of Refs. [30] and [31], and the results are summarized in Table 2. First  $\mu_c/\beta$  is required to equal its physical value of 1.12, and the input parameters are varied under the constraint that  $\mu_c/\beta$  remains constant. As the first line of Table 2 shows,  $s_q$  is

less than or equal to 0.06 under this condition. The last column of Table 2 gives the location in parameter space where the maximum value of  $s_q$  is achieved.

For the remainder of the parameter space search,  $s_q$  is fixed at some value and the input parameters vary under the condition that  $s_q$  remains constant. In each case,  $\mu_c/\beta$  is bounded from above; the bounds are listed in Table 2.

Table 2 demonstrates numerically what was said above in words: the spin content and the magnetic moments cannot both be simultaneously fitted in this model. Any kind of best fit would probably involve making both  $s_q$  and  $\mu_c/\beta$  some fraction (1/4 or less) of their experimental values. This is the main result of the paper.

## 5 Discussion

The above analysis shows that  $\mu_c/\beta$  and  $s_q$  cannot both be of  $\mathcal{O}(1)$  simultaneously. As discussed in Section 2, however, all of this analysis used the semi-classical approximation. This approximation is valid only if the rotational energy does not contribute much to the total mass; i.e., if  $m_{cl}/M_{TOT} \approx 1$ . However, for all the points listed in Table 2,  $m_{cl}/M_{TOT}$  is between 0.2 and 0.003. Therefore it is necessary to go beyond the semi-classical approximation.

In other words, the qualiton model is not a viable model of the constituent quark if the semi-classical approach is used. The approach itself is not valid in the region of parameter space where the model starts to become interesting. If a better approximation can be used, then what is the hope for the future of the qualiton model?

The qualiton still faces two obstacles: its excessive mass, and its strong interactions. First, the mass: within the semi-classical approximation the product of the mass and the rms radius exceeds a plausible value, even at its minimum. While moving in parameter space away from this minimum in a direction that favors realistic magnetic moments or spin content, the moments of inertia become so small that the semi-classical approximation is suspect. Improving the approximation in the manner suggested after Eq. (11) may increase the moments of inertia and so lower the mass, but this improvement seems unlikely to lower the mass enough to make  $Mr_{rms}$  realistically small. Another kind of



improvement on the semi-classical approximation, the inclusion of additional degrees of freedom, will only increase the mass. So it is likely that no matter what approximation is used, the mass will be larger than expected for a constituent quark. If the constituent quark is a soliton, the question is no longer “What makes the constituent quark so heavy?” but “What makes it so light?”

Second, the strong coupling: in order to make  $s_q$  large enough,  $\alpha_s$  must be large. This is because when  $\alpha_s$  is small ( $\alpha_s \lesssim 1$ ),  $s_q \propto \alpha_s$ . In the semi-classical approach, the proportionality constant is roughly  $1.7 \times 10^{-2}$ , almost independent of  $e$  and  $\nu$ . Unless this constant changes by more than two orders of magnitude when the semi-classical approach is discarded,  $\alpha_s$  will have to be at least of  $\mathcal{O}(1)$  if  $s_q$  is to be of  $\mathcal{O}(1)$ . However, any  $\alpha_s \gtrsim 1$  will sabotage the constituent model because the constituents need to be perturbatively interacting for the model to work.

There is one potential solution to this problem: the gauge field surrounding the soliton may screen the particle’s charge, so that even when  $\alpha_s$  is large, the interactions between qualitons can be treated perturbatively. Preliminary calculations indicate that some screening does occur. However it remains to be seen whether, once the qualiton is fully quantized (beyond the semi-classical approximation), this screening is enough to make the qualiton model realistic.

In short, the constituent quark cannot be described by the qualiton in the semi-classical approximation, and if a better approximation gives the correct spin and magnetic properties, the qualiton’s large mass and strong interactions will have to be explained.

Faced with these difficulties, it is tempting to return to the chiral quark model mentioned in the introduction. One may even wonder whether constituent quarks exist at all. Perhaps the constituent quark model operators and wave functions simply have the right symmetry properties, and corrections to their matrix elements are suppressed for some reason (for example, by powers of  $1/N_c$  [32]). In any case the success of the constituent quark model is not yet understood.

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## Figure Captions

**Fig. 1** The functions  $F(\tilde{r})$  and  $\gamma(\tilde{r})$  for  $1/e = 0$ ,  $\alpha_s = 12.25$ , and  $\nu = 237$ . The soliton has  $\mu_c/\beta = 2.4 \times 10^{-4}$  and  $s_q = 0.75$ .

**Fig. 2** The functions  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  for the same input parameters as in Fig. 1.

**Table 1** Quark contributions to the spin of the proton, using Eqns. (28) - (29) and EMC [23] (line 1), E142 [24] (line 2), or SMC [25] data (line 3). Line 4 gives Ellis and Karliner's analysis [26]. Line 5 gives the constituent quark model (CQM) prediction, which uses  $s_q = 1$  and  $\Delta g = 0$ . Line 6 gives the results of the CQM with the modification that  $s_q = 3/4$  and  $(\alpha_s/2\pi)\Delta g = 0.2$ .

**Table 2** The spin and magnetic properties of the soliton. The first line shows that when the input parameters are varied keeping  $\mu_c/\beta$  fixed,  $s_q$  is always less than the given bound. The rest of the table shows that for  $s_q$  fixed,  $\mu_c/\beta$  is bounded. These bounds are compared with experiment.

	$\Delta u$	$\Delta d$	$\Delta s$
EMC	$.74 \pm .06$	$-.52 \pm .06$	$-.19 \pm .06$
E142	$.93 \pm .03$	$-.33 \pm .03$	$0.00 \pm .04$
SMC	$.75 \pm .08$	$-.51 \pm .08$	$-.18 \pm .08$
Ellis & Karliner	$.80 \pm .04$	$-.46 \pm .04$	$-.13 \pm .04$
CQM	1.33	-.33	0
Modified CQM	.80	-.45	-.20

Table 1:

$s_q$	$\mu_c/\beta$	Point where upper limit is reached
$\leq .06$	1.12 (fixed)	$e = 975, \alpha_s = 56, \nu = 84$
.2 (fixed)	$\leq .08$	$e = 970, \alpha_s = 27, \nu = 88$
.4 (fixed)	$\leq .008$	$e = 834, \alpha_s = 20, \nu = 160$
.75 (fixed)	$\leq .0006$	$e = 808, \alpha_s = 13, \nu = 232$
.75	1.12	Experiment

Table 2:

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